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A Helical Wave Guide

by

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ABSTRACT

This paper contains an investigation of the field inside an idealized helical wave guide having cylindrical walls which are perfectly conducting in a helical direction and perfectly non-conducting in the direction perpendicular to this. It is found that there are in general an infinite number of non-attenuated modes of propagation along such a guide; some of these modes have phase velocities along the guide greater than the velocity of light in free space and others have smaller phase velocities along the guide. By proper choice of the design parameters of the guide it is possible to eliminate all modes having phase velocities along the guide greater than the free space velocity of light, and at the same time to separate the phase velocities of the other modes. Such a choice of parameters is desirable in the traveling wave tube application of this theory.

CONTENTS

1. Introduction. In recent years a great deal of research has been done on an ultra-high-frequency amplifier known as the traveling wave tube.* It has been found that this tube possesses the desirable properties of high gain, broad bandwidth, and low noise level. The heart of the tube is a helical transmission line which is designed to pass an electromagnetic wave with wave length along the guide, and hence phase velocity along the guide, smaller than that of the free-space wave. An electron beam, having a velocity approximately equal to the phase velocity of the wave, is shot down the center of the helical line. As a result of the interaction between the beam and the electromagnetic wave the wave amplitude is magnified.

This paper consists of a study of the transmission properties of an idealized helical wave guide. To date the traveling wave tube has been constructed with a single wire wound in the shape of a helix. Such a construction possesses the disadvantage of not being very rigid. A more rigid construction could be achieved by taking a cylindrical guide made of insulating material, threading the inside at the desired pitch, and filling the resulting valleys with the conducting material. Such a guide would be equivalent to several adjacent strands of wire each wound in the shape of a helix. If, now, the threads in the threaded cylindrical guide are made finer and finer, still maintaining the original pitch and depth, one arrives

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* The traveling wave tube was developed at Oxford University by a group under R. Kompfner. Some of the results of their research can be found in the following reports:

C.V.D.Report,C.L.Misc. 26, C.R.D.Ref. 44/3613
C.V.D.Report,C.L.Misc. 28, C.R.D.Ref. 44/3910
C.V.D.Report,C.L.Misc. 40, C.R.D.Ref. 44/1681
Wireless World, November, 1946

Dr. J. R. Pierce of the Bell Telephone Laboratories has also done a considerable amount of work in this field. His work is expected to be published in the February, 1947 issue of the Bell Telephone System Jr.

at the idealization considered in this paper.

It may be that sparking across the insulating material would make such a construction as we have proposed impractical. In any case one would like to know how close the electrical behavior of such a construction is to that of the single strand helix. It is clear that the current is constrained to travel along a helix having the same pitch in both cases. Thus the boundary conditions are somewhat the same for the two designs. However, there are air gaps between these currents in the single wire helix which are not present in the other case. This has the effect of increasing the average inner radius of the helix. Such an intuitive argument can be justified on the basis of experiment. If one applies the theory for the idealized helical wave guide to the single wire helix, one can obtain the observed phase velocity along the guide by slightly fudging the radius of the cylinder. This fudge factor depends on the spacing between the turns of wire as compared to the wave length of the transmitted wave. For a free space wave length of 82 cm., a spacing between turns of 2 cm., and a helical radius of 3 cm. (inner) and 4.3 cm. (outer), it was necessary to assume in the theory for the idealized helix a cylinder radius of 4.5 cm. in order to obtain the experimentally determined phase velocity of the single wire helix. In another instance for a free space wave length of 10 cm., a spacing between turns of 0.25 cm., and a helical radius of 2.6 mm (inner) and 3.8 mm (outer), it was necessary to assume a cylinder radius of 4 mm. in order to achieve agreement.

The precise boundary conditions for the idealized helical wave guide can be formulated as follows. The guide consists of a hollow circular cylinder which extends indefinitely far in both directions. The inner surface of the cylinder is perfectly conducting in the helical direction and perfectly

non-conducting in the direction perpendicular to the helical direction. It follows that on the cylinder surface the electrical field vanishes in the helical direction. There is some degree of arbitrariness in the magnetic field boundary conditions. We have supposed that the conducting threads in the previously mentioned limiting process are uniformly deep. In this case the magnetic field likewise vanishes along the direction of the helix. These boundary conditions lead to the existence of certain normal modes which, in the usual wave guide terminology, are linear combinations of transverse-electric and transverse-magnetic modes. For each $n (n = 0, \pm 1, \pm 2, \ldots\ldots)$ there is a solution of the wave equation in cylindrical coordinates involving n^{th} order Bessel functions and having n nodes around the circumference. For each such solution there are precisely as many non-attenuated modes as there are real solutions in u ($u \leq ak$) and imaginary solutions in u of the equation

$$\frac{\alpha}{a} \cdot \frac{1}{ak} = \frac{n}{u^2} \left[1 - \left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} \pm \frac{J_n'(u)}{u \ J_n(u)} . \quad (1)$$

Here $k = \frac{2\pi}{\lambda_0}$ where λ_0 is the free-space wave-length, a is the radius of the cylindrical guide, and $\alpha = \frac{d}{2\pi}$ where d is the distance between turns of the helix. In general a finite number of real solutions less than ak can be found for each of a finite number of the n's. In addition either one, or two imaginary solutions can be found for each $n < 0$ and either zero, one, or two imaginary solutions for each $n \leq 0$. We shall refer to a mode that corresponds to a real solution less than ak of Eq. (1) as an R-mode, and one that corresponds to an imaginary solution as an I-mode.

In the traveling wave tube it is necessary that a wave having a phase velocity less than the free space wave be propagated down the helical guide. Only I-modes have phase velocities less than the free space wave. However, an infinite number of non-attenuated I-modes can be transmitted

down any helical guide. In order that the traveling wave tube be stable it is necessary that the I-mode used be well isolated in phase velocity from any of the other possible modes. One way of accomplishing this is to use the zero-order I-mode to interact with the electron beam, and to design the helical guide so that the parameter ak is small compared to one. The zero-order I-mode will exist if an only if the parameter $\frac{\alpha}{a} \cdot \frac{1}{ak}$ is less than one-half, in which case, of course Eq.(1) must have an imaginary solution, u = iv. This solution can be read from the graph in Fig. 1. For values of $\frac{\alpha}{a} \cdot \frac{1}{ak} < 0.2$, the corresponding solution is approximately

$$v \simeq \frac{a}{\alpha} \cdot ak \tag{2}$$

The wave length along the tube for the I-modes is

$$\lambda_z = \lambda_0 \left[1 + \left(\frac{v}{ak} \right)^2 \right]^{-1/2} . \tag{3}$$

For the zero order I-mode with $\frac{\alpha}{a} \cdot \frac{1}{ak} < 0.2$, this becomes

$$\lambda_z \simeq \lambda_0 \frac{\alpha}{(a^2 + \alpha^2)^{1/2}} , \tag{4}$$

which is precisely what one would obtain if the wave followed the helical windings in the cylinder with its free space velocity.

Although the condition, $\frac{\alpha}{a} \cdot \frac{1}{ak} < \frac{1}{2}$ is sufficient to insure the existence of the zero-order I-mode, it is not sufficient to eliminate the possibility of other modes nor even to isolate the zero-order I-mode with respect to phase velocity. It is possible, however, to isolate the phase velocity of the zero-order I-mode from that of the other I-modes by choosing ak small compared to one. If one imposes these restrictions on $\frac{\alpha}{a} \cdot \frac{1}{ak}$ and ak, all of the R-modes are automatically eliminated. This

can be verified by means of Fig. 2 which defines the region containing all possible values of the two parameters for which no H-modes exist. Furthermore, if the above conditions are fulfilled, only I-modes will exist for $n \leq 0$, and for each $n<0$ there will be two imaginary solutions of Eq. (1) having the approximate values,

$$\frac{v_n}{aK} \simeq \left(\frac{|n|}{aK} - 1\right)\frac{a}{\alpha}$$

and

$$\frac{v_n}{aK} \simeq \left(\frac{|n|}{aK} + 1\right)\frac{a}{\alpha} \qquad (5)$$

Inserting this in Eq. (3), we obtain

$$\lambda_z \simeq \lambda_0 \frac{\alpha K}{|n|}\left[1 \pm \frac{aK}{|n|}\right],$$

for $\dfrac{|n|}{\alpha K} >> 1$.

This paper contains a discussion of all the non-attenuated transmitted modes for the idealized helical wave guide, special emphasis being placed on the zero-order I-mode. For this mode the field inside the guide has been computed; in addition the attenuation has been found for the case when the walls are not perfect conductors.

2. Mathematical Formulation of the Problem. For a monochromatic source the electromagnetic field inside an infinitely long circular cylinder can be represented in the form*

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* See Stratton, J.A. : Electromagnetic Theory, p. 524. McGraw-Hill.

$$E_r = \sum_{n=-\infty}^{\infty} \left[\frac{i\beta}{\gamma} J_n'(\gamma r) A_n - \frac{\mu\omega n}{\gamma^2 r} J_n(\gamma r) B_n \right] F_n$$

$$E_\theta = \sum_{n=-\infty}^{\infty} \left[-\frac{n\beta}{\gamma^2 r} J_n(\gamma r) A_n + \frac{i\mu\omega}{\gamma} J_n'(\gamma r) B_n \right] F_n$$

$$E_z = \sum_{n=-\infty}^{\infty} \left[J_n(\gamma r) A_n \right] F_n \tag{6}$$

$$H_r = \sum_{n=-\infty}^{\infty} \left[\frac{n k^2}{\mu\omega\gamma^2 r} J_n(\gamma r) A_n + \frac{i\beta}{\gamma} J_n'(\gamma r) B_n \right] F_n$$

$$H_\theta = \sum_{n=-\infty}^{\infty} \left[\frac{i k^2}{\mu\omega\gamma} J_n'(\gamma r) A_n - \frac{n\beta}{\gamma^2 r} J_n(\gamma r) B_n \right] F_n$$

$$H_z = \sum_{n=-\infty}^{\infty} \left[J_n(\gamma r) B_n \right] F_n .$$

In these relations

$$\gamma^2 = k^2 - \beta^2 ,$$

$$F_n = e^{in\theta + i\beta z - i\omega t} \tag{7}$$

and the prime above a Bessel function denotes differentiation with respect to the argument γr. Without loss of generality we shall limit our considerations to β with real part greater than or equal to zero.

It is assumed for the idealized helical wave guide that on the inner surface of the cylinder both the electric field and the magnetic field will vanish in the helical direction. The helical direction at the point (a, θ, z) can be designated as

$$\underline{s} = 0\, \underline{v_r} + a\, \underline{v_\theta} + \alpha\, \underline{v_z}$$

where, as before, a is the radius of the cylindrical guide and

$\alpha = d/2\pi$, d being the distance between turns on the helix. The boundary conditions can then be expressed as

$$\underline{E} \cdot \underline{n} = a E_\theta + \alpha E_z = 0 \qquad (8)$$
$$\underline{H} \cdot \underline{n} = a H_\theta + \alpha H_z = 0$$

for r = a and all values of θ and z.

Since the equations (8) are valid for all values of θ, they are also valid for the Fourier coefficients of these quantities taken with respect to θ; that is the corresponding bracketed expressions in Eq. (6) times $e^{i\beta z - i\omega t}$. Finally since these relations hold for all z, they likewise hold for the bracketed terms alone. In this form, the boundary conditions are

$$\left(\alpha - \frac{n\beta}{\gamma^2}\right) J_n(\gamma a)\, A_n - \frac{i\mu\omega a}{\gamma} J_n'(\gamma a)\, B_n = 0. \qquad (9)$$
$$\frac{i K^2 a}{\mu\omega\gamma} J_n'(\gamma a)\, A_n + \left(\alpha - \frac{n\beta}{\gamma^2}\right) J_n(\gamma a)\, B_n = 0.$$

In order that there exist a non-trivial solution for A_n, B_n, the determinant of coefficients of Eq. (9) must vanish; that is

$$\left(\alpha - \frac{n\beta}{\gamma^2}\right)^2 J_n^2(\gamma a) = \frac{K^2 a^2}{\gamma^2}\left[J_n'(\gamma a)\right]^2,$$

or

$$\frac{\alpha}{a} \cdot \frac{1}{aK} = \frac{n\beta}{(\gamma a)^2 K} \pm \frac{J_n'(\gamma a)}{\gamma a\, J_n(\gamma a)}. \qquad (10)$$

To each solution γ of Eq. (10), there corresponds a natural mode of the helical wave guide which can be propagated (with or without attenuation) down the helix. Substituting in Eqs. (9) and (6) one finds that the field equations for this mode are

$$E_r = \left[\mp \frac{\beta}{\gamma} J_n'(\gamma r) - \frac{Kn}{\gamma^2 r} J_n(\gamma r)\right]\eta \quad F_n$$
$$E_\theta = \left[\mp \frac{i n\beta}{\gamma^2 r} J_n(\gamma r) - \frac{iK}{\gamma} J_n'(\gamma r)\right]\eta \quad F_n$$
$$E_z = \pm i J_n(\gamma r)\,\eta\, F_n \qquad (11)$$

$$H_n = \left[\pm \frac{ink}{\gamma^2 r} J_n(\gamma r) + \frac{i\beta}{\gamma} J_n'(\gamma r) \right] F_n$$

$$H_\theta = \left[\mp \frac{k}{\gamma} J_n'(\gamma r) - \frac{n\beta}{\gamma^2 r} J_n(\gamma r) \right] F_n \qquad (11)$$

$$E_z = \left[J_n(\gamma r) \right] F_n ,$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the resistance of the medium in ohms. Here the upper sign refers to a solution of Eq. (10) with a plus sign, whereas the lower sign refers to a minus-sign solution of Eq.(10). These field equations correspond to linear combinations of the familiar transverse-electric and transverse-magnetic modes present in ordinary wave guides. This is not the only novel feature in these field equations. If we consider only β with positive real part, the relation (10) is essentially different for positive and negative values of n. Hence, unlike the usual wave guide, the modes propagating down the guide which can exist for positive n are different from those which can exist for negative n. This, of course, should be expected in view of the asymmetry of the boundary conditions with respect to θ .

In order that a mode be propagated down the guide without attenuation it is necessary that β be real valued. By Eq. (7)

$$\beta = \sqrt{k^2 - \gamma^2} \quad .$$

Hence β will be real valued if and only if γ is either imaginary valued or real valued and less than k. As mentioned in section 1, the non-attenuated modes have been divided into two classes; the I-modes corresponding to imaginary solutions of Eq. (10); and the R-modes corresponding to real γ- solutions ($\gamma \leq k$) of Eq. (10). For the I-modes β is greater than k so that the phase velocity along the axis is less than the velocity of light in free space; whereas for the R-modes the phase velocity along the axis is more than the velocity of light in free space.

For $n \neq 0$ it is possible to find real solutions of Eq. (10) even for $\gamma > k$. For the corresponding modes β is pure imaginary and consequently the mode is attenuated. For $n = 0$, however, Eq. (10) has no real valued solutions, $\gamma > k$, because for such γ the β term is the only complex valued term in the equation. Since from physical considerations it seems clear that fairly general boundary conditions can be matched by each set of modes for a given n, it follows that there must be an infinite set of solutions for each n. However, as we shall show, there are only a finite number of R- and I- modes for each n. Consequently there must exist complex values of γ satisfying Eq. (10), for all $n \neq 0$. The corresponding β is also complex valued so that the mode behaves like an attenuated wave even though there is no dissipation of energy along the guide*. Such modes do not occur in the usual perfect conductor wave guide.

The remainder of this paper will be devoted to a discussion of the R- and I- modes.

3. The Zero-Order Modes. In the traveling wave tube, the zero-order I-mode has been used to achieve the desired results. For this reason and because the zero-order modes are somewhat typical of the other modes, the present section will be devoted to a discussion of these modes. The section will be descriptive since the zero-order modes are but a special case of the general n^{th}-order modes which will be studied in later sections.

For $n = 0$, Eq. (10) becomes

$$\frac{\alpha}{a}\,\frac{1}{\epsilon k} = \pm\,\frac{J_0'(u)}{uJ_0(u)} \qquad (12)$$

where $u = a\gamma$. It is convenient to introduce the function

$$g_0(u) = \frac{J_0'(u)}{uJ_0(u)}\ .$$

*The boundary conditions along the guide are such that the component of the Poynting vector normal to the guide vanishes. It follows that no energy leaks out of the guide. Hence for damped out waves there cannot be a net flow of energy in the direction of the guide.

It is clear that any value of u, for which $z_0(u)$ is equal to either $\frac{\alpha}{a}\cdot\frac{1}{aK}$ or $-\frac{\alpha}{a}\cdot\frac{1}{aK}$, is a solution of Eq. (12).

We shall first consider the I-modes. For this case let $u = iv$; v is then real valued. z_0 is a negative monotonically increasing function of v for $v \geq 0$. The function $-z_0$ is plotted as a function of v in Fig. 1. The series expansion for z_0 about the point $v = 0$ has the form

$$z_0 = -\frac{1}{2} + \frac{v^2}{16} - \frac{v^4}{96} + \dots .$$

On the other hand the asymptotic expansion for z_0 is of the form

$$z_0 = -\frac{1}{v} + \frac{1}{2}\left(\frac{1}{v}\right)^2 + \frac{1}{8}\left(\frac{1}{v}\right)^3 + \dots .$$

It follows that a zero-order I-mode will exist if and only if

$$0 < \frac{\alpha}{a}\cdot\frac{1}{aK} \leq \frac{1}{2} \; ; \tag{13}$$

and when this is the case there is a unique solution for Eq. (12). As can be seen from the asymptotic expansion, when $\frac{\alpha}{a}\cdot\frac{1}{aK}$ is small, say less than 0.2, this solution is given approximately by Eq. (2).

The field equations for the zero-order I-mode are simply

$$E_r = \frac{\beta}{\gamma} J_0'(\gamma r)\,\eta\, F_0, \qquad H_r = \frac{i\beta}{\gamma} J_0'(\gamma r)\, F_0 ,$$

$$E_\theta = -\frac{iK}{\gamma} J_0'(\gamma r)\,\eta\, F_0, \qquad H_\theta = \frac{K}{\gamma} J_0'(\gamma r)\, F_0 ,$$

$$E_z = -i\, J_0(\gamma r)\,\eta\, F_0, \qquad H_z = J_0(\gamma r)\, F_0 .$$

The factor i can be interpreted as a phase advance of a quarter of a wave length (i.e. $\lambda_z/4$) in the z-direction. Hence the E and the H fields differ only in the factor η and the fact that E lags H by a quarter of a wave length in the z-direction. That is, for the zero-order I-mode

$$\underline{E}(r,\theta,z) = \eta\, \underline{H}(r,\theta, z - \lambda_z/4).$$

Sectional drawings of the zero-order I-mode E-field are shown in Fig. 3. The parameters for this particular helical wave guide are

$$\frac{\alpha}{2}\cdot\frac{1}{2k} = \frac{1}{4}\ ; \quad 2k = \frac{1}{7}$$

The helical windings form a left-hand screw thread for which $d/a = 2\pi/16 = 0.39$. As can easily be computed from Fig. 2 and Eq. (7),

$$a\gamma = 3.351 \qquad \text{and } \beta/k = \frac{\lambda_0}{\lambda_z} = 13.4 \ .$$

The figure shows the E-field for only half a wave length since the remaining half is the same except for sign.

In an actual helical wave guide of the type considered in this paper, the walls of the guide would not be perfect conductors in the helical direction. If the conductivity in this direction is finite but large, one can approximate the attenuation along the guide rather simply.* One assumes that the field pattern at a given cross section is essentially the same for a guide that is perfectly conducting and for one that is not. In the neighborhood of this cross section one can then calculate both the energy flow across the section and the heat dissipation in the walls. The attenuation factor is then

$$\text{Attenuation factor} = \frac{1}{2}\,\frac{\text{Power loss per unit length}}{\text{Power transfer along guide}} \ . \qquad (14)$$

For the zero-order I-mode the power transfer along the guide is the integral of the z-component of the Poynting vector across the cross section. Now the z-component of the Poynting vector is

$$(E \times H)_z = \frac{\beta k}{\gamma^2}\eta\left[J_0'(\gamma r)\right]^2 \ .$$

* See, for instance, Ramo and Whinnery, Fields and Waves in Modern Radio, Chapter 8, 1944, Wiley and Sons.

As can be seen, the energy flow per second depends only on r and is the same for all cross sections, all time, and all θ. The energy flow through any cross section is then

$$\int_0^a r\,dr \int_0^{2\pi} d\theta\,(E \times H)_z = \pi a^2 \frac{\beta k}{\gamma^2 \eta} J_1^2(\gamma a)\left[1 + \left(\frac{ak}{\alpha\gamma}\right)^2 - \frac{2\alpha k}{(\alpha\gamma)^2}\right]. \quad (15)$$

On the other hand, the average RI loss per meter is the time-average of $I^2 R_s \, 2\pi a$. Here I is equal to the tangential component of H at the walls of the guide and R_s is the effective resistance in ohms per square meter of the walls. At r = a the tangential component of H is of course perpendicular to the helical direction; that is $aH_\theta \quad H_z = 0$. Hence

$$I^2 = \left[1 + \left(\frac{\alpha}{a}\right)^2\right] H_z^2 = \left(1 + \frac{\alpha}{a}\right)^2 J_0^2(\gamma a)\cos^2(\beta z - \omega t).$$

For plane conducting surfaces, or where the radius of curvature is of the same magnitude or larger than the wave length,

$$R_s = \sqrt{\frac{\omega \mu}{2\sigma}}$$

where σ is the conductivity. The average RI loss per meter is readily seen to be

$$\text{averaged RI loss per meter} = \pi R_s \frac{a^2 + \alpha^2}{a} J_0^2(\gamma a). \quad (16)$$

Combining Eqs. (14), (15), and (16), one obtains

$$\text{Attenuation factor} = \frac{R_s (a^2 + \alpha^2) k}{2\eta\, \alpha^2 a \beta} \cdot \frac{1}{\left[1 + \left(\frac{ak}{\alpha\gamma}\right) - \frac{2\alpha k}{(\alpha\gamma)^2}\right]}.$$

Before concluding this section we shall consider the E-modes; these are the modes associated with the real solutions of Eq. (12). A graph of z_0 as a function of u is shown in Fig. 4. This function starts at -1/2, has poles of the first order at all of the zeros of $J_0(u)$, and is everywhere monotonically decreasing. The series expansion for z_0 about u = 0 is

$$z_0 = -\frac{1}{2} - \frac{u^2}{16} - \frac{u^4}{96} - - - - ;$$

for large values of u, z_0 behaves like

$$-\frac{1}{u}\tan\left(u - \pi/4\right) .$$

In general there will be many u-solutions of Eq. (12) smaller in value than ak. If ak is less than the first zero, $x_1^0 = 2.405$, of $J_0(u)$, then any E-mode solution must involve only the first branch of the z_0 curve. As can easily be seen in Fig. 4, there will be no solution if $\frac{\alpha}{a}\cdot\frac{1}{ak}$ is entirely above this branch or if it lies below that part of the branch for which $u \leq ak$. In other words, if $ak < x_1^0$, there can be no E-mode solution if

$$\frac{\alpha}{a}\cdot\frac{1}{ak} < \frac{1}{2} \tag{17a}$$

or if

$$\frac{\alpha}{a}\cdot\frac{1}{ak} > -z_0(ak). \tag{17b}$$

similarly if ak is simply less than the first zero $x_1^1 = 3.83$ of $J_0'(u) = -J_1(u)$ there can exist no zero-order E-mode if

$$\frac{\alpha}{a}\cdot\frac{1}{ak} < \min\left[\frac{1}{2}, \left|z_0(ak)\right|\right]. \tag{17c}$$

There will, however, be an infinite set of u's for which Eq. (12) is satisfied; these will all be values of u greater than ak and therefore correspond to attenuated modes. If condition (17c) is satisfied, then the zero-order I-mode can exist but no zero-order E-mode can exist.

4. Restatement of the Problem. To each solution of Eq. (10), there corresponds a natural mode of the helical wave guide. We shall consider only the non-attenuated modes; that is real $\gamma < k$ (R-modes) and pure imaginary γ (I-modes). In either case β will be the positive square root of $(k^2 - \gamma^2)$. Since $J_{-n}(u) = (-1)^n \; J_n(u)$, we can replace $\frac{J'_{-n}(u)}{uJ_{-n}(u)}$ by $\frac{J'_n(u)}{uJ_n(u)}$. This permits us to consider the solutions of Eq. (10) for positive and negative values of n simultaneously by finding the solutions of

$$\pm \frac{\alpha}{a} \cdot \frac{1}{ak} = \frac{n\beta}{u^2 k} \pm \frac{J'_n(u)}{uJ_n(u)} \quad (n = 0,1,2,\ldots) \quad (18)$$

where $a\gamma$ has been replaced by u. Once a solution is obtained, it will of course be easy to determine whether it corresponds to n or to -n. If the plus sign holds on the left then it will correspond to n, whereas if the minus sign holds on the left it will correspond to -n.

The non-attenuated solutions of Eq. (18) can be found by studying the four functions on the right-hand side of the equation defined by the plus and minus signs and the real and pure imaginary arguments. More explicitly these functions are

$$
\begin{aligned}
I_{1,n}(v) &= -\frac{n}{v^2}\left[1+\left(\frac{v}{ak}\right)^2\right]^{\frac{1}{2}} + \frac{J'_n(iv)}{ivJ_n(iv)}, &&(n = 0,1,2,\ldots),\\
I_{2,n}(v) &= -\frac{n}{v^2}\left[1+\left(\frac{v}{ak}\right)^2\right]^{\frac{1}{2}} - \frac{J'_n(iv)}{ivJ_n(iv)}, &&(n = 0,1,2,\ldots),\\
R_{1,n}(u) &= \frac{n}{u^2}\left[1-\left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} + \frac{J'_n(u)}{uJ_n(u)}, &&(n = 0,1,2,\ldots),\\
R_{2,n}(u) &= \frac{n}{u^2}\left[1-\left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} - \frac{J'_n(u)}{uJ_n(u)}, &&(n = 0,1,2,\ldots).
\end{aligned}
\quad (19)
$$

These functions will be treated in turn.

The principal tool used in the investigation of these functions is the differential equation for

$$z_n = \frac{J_n'(u)}{uJ_n(u)} \quad . \tag{20}$$

By differentiating z_n and making use of the defining differential equation for Bessel functions, one readily obtains

$$\frac{dz_n}{du} = -uz_n^2 - \frac{2}{u} z_n - \frac{1}{u} + \frac{n^2}{u^3} \quad , \tag{21a}$$

a differential equation of the Ricatti type.

Replacing u by iv, the equation becomes

$$\frac{dz_n}{dv} = vz_n^2 - \frac{2}{v} z_n - \frac{1}{v} - \frac{n^2}{v^3} \quad . \tag{21b}$$

One can obtain a series expansion for z_n by dividing the corresponding Bessel function expansions. This gives

$$z_n = \frac{-n}{u^2}\left[1 - \frac{2}{n(n+1)}\left(\frac{u}{2}\right)^2 - \frac{2}{n(n+1)^2(n+2)}\left(\frac{u}{2}\right)^4 - - - -\right] \tag{22a}$$

and

$$z_n = -\frac{n}{v^2}\left[1 + \frac{2}{n(n+1)}\left(\frac{v}{2}\right)^2 - \frac{2}{n(n+1)^2(n+2)}\left(\frac{v}{2}\right)^4 - - - -\right] \tag{22b}$$

One can likewise obtain the asymptotic expansions for z_n. To avoid differentiating an asymptotic expansion, we replace J_n' by $\frac{1}{2}(J_{n-1} - J_{n+1})$. This yields

$$z_n = -\frac{1}{u}\tan\left[u - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right] \tag{23a}$$

and

$$z_n = -\frac{1}{v} + \frac{1}{2}\left(\frac{1}{v}\right)^2 - \frac{1}{2}\left(n^2 - \frac{1}{4}\right)\left(\frac{1}{v}\right)^3 - \frac{1}{2}\left(n^2 - \frac{1}{4}\right)\left(\frac{1}{v}\right)^4 - - - \tag{23b}$$

5. The I-modes. The principal result of the present section is the fact that an infinite set of I-modes are always present. Aside from the zero-order mode, the $i_{1,n}(v)$ functions start out with a pole at the origin, are always negative and monotonic increasing, and approach the real axis asymptotically as $v \to \infty$. Hence to each such function there corresponds a solution of Eq. (18) for each negative n. The $i_{2,n}(v)$ functions are somewhat more complicated and may have zero, one, or two solutions for positive n, and zero or one solution for negative n. In this section the independent argument will always be $v = -iu$.

Lemma 1. z_n is a negative monotonically increasing function of v for all $v > 0$.

As can be seen from the series expansion, Eq. (22b), z_n is negative valued for sufficiently small v. From known properties of Bessel functions z_n is an analytic function for $v > 0$. Consequently if it ever attains a positive value for any $v > 0$ it must first cross the v-axis; that is it must vanish at some point, say $v = v_1 > 0$. But by Eq. (21b).

$$\left.\frac{dz_n}{dv}\right)_{v_1} = -\frac{1}{v_1} - \frac{n^2}{v_1^3} < 0.$$

That is z_n is a decreasing function of v at $v = v_1$. In other words, at the point where z_n crosses the v-axis, z_n must be changing from positive to negative values with increasing v. This is contrary to the fact that z_n is negative-valued for small values of v. It follows that $z_n < 0$ for all $v > 0$.

On the other hand, as can be seen from the series expansion, $\frac{dz_n}{dv}$ is positive for sufficiently small $v > 0$. Differentiating Eq. (21b),

one obtains

$$\frac{d^2 z_n}{dv^2} = (2vz_n - \frac{2}{v}) \frac{dz_n}{dv} + z_n^2 + \frac{2}{v^2} z_n + \frac{1}{v^2} + \frac{3n^2}{v^4} . \quad (24)$$

Setting $\frac{dz_n}{dv} = 0$ and combining Eqs. (21b) and (24), we have

$\frac{d^2 z_n}{dv^2} = 2z^2 + \frac{2n^2}{v^4} > 0.$ This means that if $\frac{dz_n}{dv}$ were to vanish for any value of v, it would be changing from negative to positive values. It follows that $\frac{dz_n}{dv}$ must remain positive for all $v > 0$.

We can also say something about the relative values of z_n for different values of n, namely the following lemma.

Lemma 2. $z_n > z_{n+1}$ for all $n \geq 0$ and $v > 0$.

Let $\Delta z = z_n - z_{n+1}$. It follows from the series expansion Eq. (22b), that for small v, Δz behaves like $1/v^2$. By Eq. (21b),

$$\frac{d(\Delta z)}{dv} = v(z_n + z_{n+1}) \Delta z - \frac{2}{v} \Delta z + \frac{2n+1}{v^3} . \quad (25)$$

Again, Δz is positive for small values of v, and its graph can never cross the v-axis, since if Δz should ever vanish, $\frac{d \Delta z}{dv} = \frac{2n+1}{v^3} > 0$ (for $v > 0$).

We see therefore that the z_n's are negative monotonically increasing functions which form an ordered family, the graph of z_n lying completely above that of z_{n+1}. The $I_{1,n}$ functions have these same properties, as will be shown in Theorems 1 and 2. One readily obtains the series expansion and the asymptotic expansion of $I_{1,n}$ from Eqs. (22b) and (23b) and the definition of $I_{1,n}$ (Eq. (19)). These are

$$I_{1,n}(v) = \frac{2n}{v^2} - \left(\frac{1}{2(n+1)} + \frac{n}{2(nk)^2} \right) - - - - \quad (26)$$

and

$$I_{1,n}(v) = - \left(1 + \frac{n}{nk}\right) \cdot \frac{1}{v} + \frac{1}{2} \cdot \left(\frac{1}{v}\right)^2 - - - - - .$$

Thus, $I_{1,n}(v)$ has a pole at the origin, and approaches the v-axis asymptotically from below as $v \rightarrow \infty$.

Theorem 1. $I_{1,n}$ is a negative monotonically increasing function of v for all $v > 0$.

This follows from the fact that $I_{1,n}$ is the sum of two negative monotonically increasing functions of v.

If we now define ΔI_1 by the equation,

$$\Delta I_1 = I_{1,n} - I_{1,n+1} \;,$$

then

$$\Delta I_1 = \frac{1}{v^2}\left[1 + \left(\frac{v}{ak}\right)^2\right]^{\frac{1}{2}} + \Delta z. \tag{27}$$

ΔI_1 is therefore the sum of two positive functions, which proves

Theorem 2. $I_{1,n} \quad I_{1,n+1}$ for all $n \geq 0$ and $v > 0$.

We see that the equation

$$\pm \frac{\alpha}{n} \frac{1}{ak} = I_{1,n} \tag{18a}$$

has no solution for the plus sign, corresponding to positive n. For $n = 0$ there is one and only one solution if and only if $\frac{\alpha}{n} \cdot \frac{1}{ak} < \frac{1}{2}$. Finally, for $n < 0$, in view of Theorem 1 and the fact that $I_{1,n}$ has a pole at the origin and approaches 0 as v tends to ∞, there is always one and only one solution, v_n. It follows from Theorem 2 that $v_n < v_{n+1}$ for $n \leq 0$. (It should be recalled that the condition $n \leq 0$ corresponds to the equation $I_{1,n} = -\frac{\alpha}{n}\frac{1}{ak}$ with $n \geq 0$.) Furthermore, as can be seen from Eq. (27) the solution values of v separate out for ak small relative to one. One obtains an approximate value for these solutions valid for $\frac{\alpha}{n} \cdot \frac{1}{ak} << 1$ from the asymptotic expansion in Eq. (26). This gives

$$v_n \simeq \frac{n}{\alpha}\; ak\left(1 + \frac{n}{ak}\right) .$$

The corresponding wave numbers are

$$\beta_n \simeq k \left[1 + \left(1 + \frac{n}{ak}\right)^2 \left(\frac{n}{\alpha}\right)^2 \right]^{\frac{1}{2}} .$$

The functions $I_{2,n}$ are more complicated than the $I_{1,n}$ functions. They are not necessarily of one sign nor monotonic. Nevertheless, as we shall show, these functions are either monotonic increasing, monotonic decreasing, or have a single maximum, depending on the values of n and ak. Furthermore for each value of ak, $I_{2,n}(v) > I_{2,n\,1}(v)$ for all $v \geq 0$.

The series expansion and the asymptotic expansion of $I_{2,n}$ are

$$I_{2,n}(v) = \left(\frac{1}{2(n+1)} - \frac{n}{2(ak)^2} \right) - \left(\frac{1}{8(n+1)^2(n+2)} - \frac{n}{8(ak)^4} \right) v^2 \; - - -$$

$$I_{2,n}(v) \approx \left(1 - \frac{n}{ak}\right) \frac{1}{v} - \frac{1}{2} \left(\frac{1}{v}\right)^2 - - - - \qquad (28)$$

The poles of $\frac{n}{v^2} \frac{\beta}{k}$ and s_n, as seen in Eq. 19 and 22b, cancel each other, leaving $I_{2,n}$ regular at the origin. From these expansions it is readily seen that as $v \to \infty$, the graph of $I_{2,n}$ approaches the v-axis from below if $ak \leq n$ and from above if $ak > n$. Furthermore the slope for sufficiently small $v > 0$ can be seen to be positive for $ak < \sqrt[4]{n\,(n+1)^2\,(n+2)}$ and negative otherwise. These remarks are consistent with the following theorem.

<u>Theorem 3</u>. If $ak \leq n$, then $I_{2,n}$ is negative and monotonic increasing; if $n < ak < \sqrt[4]{n\,(n+1)^2\,(n+2)}$, then $I_{2,n}$ has exactly one maximum and no minima; and if $\sqrt[4]{n\,(n+1)^2\,(n+2)} \leq ak$ then $I_{2,n}$ is positive and monotonic decreasing. This theorem is valid for all $n \geq 0$ and $v \geq 0$.

The differential equation for $I_{2,n}$ can be derived from Eqs. (19) and (21b).

$$\frac{dI_{2,n}}{dv} = - vI_{2,n}^2 - \frac{2}{v} \left\{ n \left[1 + \left(\frac{v}{ak}\right)^2 \right]^{\frac{1}{2}} + 1 \right\} I_{2,n} + \frac{1}{v} - \frac{n^2}{v} \frac{1}{(ak)^2}$$

$$- \frac{n}{v} \cdot \frac{1}{(ak)^2} \frac{1}{\left[1 + \left(\frac{v}{ak}\right)^2\right]^{1/2}} . \qquad (29)$$

We shall first study the vector field determined by this differential equation. Setting $\frac{dI_{2,n}}{dv}$ equal to zero results in a quadratic equation in $I_{2,n}$. The roots of this quadratic equation are two functions of v. Let U(v) be the greater of these roots and L(v) the smaller. The curves representing these two functions divide the half plane $v \geq 0$ into three regions: The region above the U-curve in which $\frac{dI_{2,n}}{dv}$ is negative, the region between the U-curve and the L-curve in which $\frac{dI_{2,n}}{dv}$ is positive, and the region below the L-curve in which $\frac{dI_{2,n}}{dv}$ is again negative. (See Fig.6).

The explicit expressions for U and L are

$$U,L = \frac{1}{v}\left[-\frac{1}{v}(nx+1) \pm \sqrt{1+\frac{n^2+1}{v^2}+\frac{n}{x}\left[\frac{1}{(ak)^2}+\frac{2}{v^2}\right]}\right]. \qquad (30)$$

where $x = \left[1+\left(\frac{v}{ak}\right)^2\right]^{\frac{1}{2}}$; the plus sign goes with U, and the minus sign goes with L. It is clear from Eq.(30) that U and L are always real valued so that their graphs actually do divide the right half-plane into the three regions described in the previous paragraph.

One sees from Eq.(30) that L(v) is the sum of negative monotonically increasing terms and hence is itself negative monotonic increasing. Furthermore L(v) has a pole at the origin. It follows that the graph of $I_{2,n}(v)$ starts out above that of L(v). If it should ever touch the L(v)-curve it would cross (having a zero slope at this point) and enter a region of negative slope. Furthermore it could never again intersect that L(v)-curve, since L(v) is a monotonic increasing function. The value of $I_{2,n}(v)$ at this point of intersection would thereafter be an upper bound for $I_{2,n}(v)$. Since this value is necessarily less than zero, this would be contrary to the fact that $I_{2,n}(v)$ approaches zero asymptotically as $v \to \infty$. Consequently the graph of $I_{2,n}(v)$ lies above that of L(v) for all $v \geq 0$.

The quadratic equation in $I_{2,n}$ obtained by setting $\frac{dI_{2,n}}{dv}$ equal to zero in Eq. (29) can be rewritten as a cubic in x. The result is

$$F(x) = (ak)^2 \, I_{2,n}^2 \, x^3 + 2nI_{2,n} \, x^2 - \left[(ak)^2 \, I_{2,n}^2 + 1 - 2I_{2,n} - \left(\frac{n}{ak}\right)^2\right] x + \frac{n}{(ak)^2} = 0 \tag{31}$$

To a given value of $I_{2,n}$ and a root $x \geq 1$, there is a $v \geq 0$ such that either $U(v) = I_{2,n}$ or $L(v) = I_{2,n}$. This value of $I_{2,n}$ need not, of course, be assumed by the function $I_{2,n}(v)$. Since $F(-\infty) = -\infty$, $F(\infty) = \infty$, and $F(0) = \frac{n}{(ak)^2}$, it follows that $F(x)$ has at most two roots, $x \geq 1$. Since $L(v)$ takes in all negative values, it follows that for values of $I_{2,n} < 0$, one of these roots necessarily corresponds to a point on $L(v)$. Hence in the region that $U(v)$ is negative it must be monotonic. On the other hand for positive $I_{2,n}$, either zero, one, or two roots can lie on $U(v)$. Hence in the region that $U(v)$ is positive it is either monotonic, or has a single maximum and no minimum (since $U(v)$ approaches the v-axis asymptotically as $v \to \infty$).

To complete our description of $U(v)$ we make use of the series and asymptotic expansions of $U(v)$ which are readily obtainable from Eq. (30).

$$U(v) = \left[\frac{1}{2(n+1)} - \frac{n}{2(ak)^2}\right] - \left[\frac{1}{8(n+1)^3} - \frac{n(n+2)}{8(n+1)(ak)^4}\right] v^2$$

$$+ \left[\frac{1}{16(n+1)^5} - \frac{n}{16(n+1)^3(ak)^4} - \frac{n(n+3)}{16(n+1)\,(ak)^6}\right] v^4 \; - - - . \tag{32}$$

$$U(v) = \left(1 - \frac{n}{ak}\right) \cdot \frac{1}{v} - \left(1 - \frac{n}{2(ak)}\right) \cdot \frac{1}{v^2} + - - - .$$

For $ak \leq n$, $U(v)$ is negative for v near 0 and also for $v \rightarrow \infty$, and hence is always negative and monotonic increasing. For $n < ak < \sqrt[4]{n\,(n+1)^2\,(n+2)}$, $U(v)$ has a positive slope for small values of v and is positive for $v \rightarrow \infty$, which implies that it has a single maximum and no minima. Finally for $ak \geq \sqrt[4]{n\,(n+1)^2\,(n+2)}$, $U(v)$ has a negative slope for small values of v and is positive for $v \rightarrow \infty$, and therefore is always positive and monotonic decreasing.

We now return to the function $I_{2,n}(v)$. A comparison of the series expansions for $I_{2,n}(v)$ and $U(v)$ (Eqs.(27) and (32)) shows that the graph of $U(v)$ starts out above that of $I_{2,n}(v)$ for $ak < \sqrt[4]{n(n+1)^2\,(n+2)}$ and below for $ak > \sqrt[4]{n(n+1)^2\,(n+2)}$. Now $I_{2,n}(v)$ cannot intersect $U(v)$ from below at a point at which $U(v)$ is increasing because at such a point the slope of $I_{2,n}(v)$ would be zero (by definition of $U(v)$), so that $I_{2,n}(v)$ could only cross $U(v)$ from above. For $ak \leq n$, $U(v)$ is always monotonic increasing so that $I_{2,n}(v)$ remains between L and U, that is in the positive slope region. Consequently in this case $I_{2,n}(v)$ is negative and monotonic increasing.

For $n < ak < \sqrt[4]{n\,(n+1)^2\,(n+2)}$, $I_{2,n}(v)$ can intersect $U(v)$ only at a point at which $U(v)$ has a non-positive slope. At this point it must necessarily cross to the region above $U(v)$ since $U(v)$ is either decreasing or at its maximum whereas $I_{2,n}(v)$ has a zero slope and can only decrease by crossing over to the region above $U(v)$. It is clear that $I_{2,n}(v)$ must cross into this region since it approaches the v-axis from above and hence eventually is decreasing. Once in the region above $U(v)$, $I_{2,n}(v)$ must remain there for all larger v since $U(v)$ is thereafter a

monotonic decreasing function and $I_{2,n}(v)$ would necessarily have a zero slope at any point of intersection. Thus $I_{2,n}(v)$ starts out increasing, has its maximum at the point of intersection with $U(v)$, and is thereafter decreasing, approaching the v-axis asymptotically from above.

Finally for $ak > \sqrt[4]{n(n+1)^2(n+2)}$, $I_{2,n}(v)$ starts out above $U(v)$, and since $U(v)$ is monotonic decreasing for $v > 0$, it follows as above that $I_{2,n}(v)$ must remain above $U(v)$ for all $v > 0$. Hence in this case $I_{2,n}(v)$ is positive and monotonic decreasing. This is likewise true when the inequality is replaced by an equality since the functions $I_{2,n}(v)$ are continuous in ak. This concludes the proof of theorem 3.

We shall next show that the functions $I_{2,n}(v)$ form an ordered family of functions. This result will be an immediate consequence of the following lemmas on the function

$$w_n = - \frac{n}{v^2} - z_n .$$

This function is equal to $-z_n$ minus its principal part at the origin. The differential equation which w_n satisfies is

$$\frac{dw_n}{dv} = -vw_n^2 - \frac{2(n+1)}{v} w_n + \frac{1}{v} . \qquad (33)$$

The expansions for w_n are

$$w_n = \frac{1}{2(n+1)} - \frac{1}{8(n+1)^2(n+2)} v^2 - - -$$

$$w_n = \frac{1}{v} - \frac{1}{2}\left(\frac{1}{v}\right)^2 - - - - \qquad (34)$$

Lemma 3. w_n is a positive monotonically decreasing function of v for all $v \geq 0$.

The proof of this lemma is very similar to that of Lemma 1. It is clear that the graph of w_n starts out positive. Further, it can never cross the v-axis since if it were zero, then by Eq. (33) $\frac{dw_n}{dv} = \frac{1}{v} > 0$. Similarly, $\frac{dw_n}{dv}$ starts out negative and it, too, cannot become zero since, if it were to do so, one can easily show that $\frac{d^2 w_n}{dv^2} = -2w_n^2 < 0$.

<u>Lemma 4.</u> $w_n > w_{n+1}$ for all $n \geq 0$ and $v \geq 0$.

Let $\Delta w = w_n - w_{n+1}$. It follows from Eq. (34) that for $v = 0$, $\Delta w = \frac{1}{2(n+1)(n+2)}$. Hence Δw is positive for small values of v. Using Eq. (33) one can write down the following differential equation for Δw:

$$\frac{d\Delta w}{dv} = -v\,(w_n + w_{n+1})\,\Delta w - \frac{2(n+2)}{v}\,\Delta w + \frac{2}{v}\,w_n \,.$$

Again Δw does not become zero since if it were ever to do so then at this point $\frac{d\Delta w}{dv} = \frac{2}{v}\,w_n > 0$ by lemma 3.

<u>Theorem 4.</u> $I_{2,n} > I_{2,n+1}$ for all $n \geq 0$ and $v \geq 0$.

Let $\Delta I_2 = I_{2,n} - I_{2,n+1}$. Then, from Eq. (19) and the definitions of Δz and Δw, ΔI_2 can be written

$$\Delta I_2 = \frac{1}{v^2}\left\{\left[1 + \left(\frac{v}{nk}\right)^2\right]^{\frac{1}{2}} - 1\right\} + \Delta w. \tag{35}$$

Hence ΔI_2 is the sum of two positive functions, which proves the theorem.

As a consequence of Theorem 3 and the series expansion, Eq.(28), the equation

$$\pm\frac{\alpha}{n}\;\frac{1}{nk} = I_{2,n} \tag{18b}$$

can be characterized as follows. For the plus sign (corresponding to positive n I-modes) Eq.(18b) has no solution if $nk \leq n$, it may have two solutions for

sufficiently small $\frac{\alpha}{a}\ \frac{1}{ak}$ if $n < ak < \sqrt[4]{n\,(n+1)^2\,(n+2)}$, and it will have one solution for $ak > \sqrt[4]{n\,(n+1)^2\,(n+2)}$ ~~and~~ if and only if

$$0 < \frac{\alpha}{a}\ \frac{1}{ak} < \frac{1}{2(n+1)} - \frac{n}{2(ak)^2} .$$

On the other hand it will have one solution for the minus sign (corresponding to negative n I-modes) if and only if

$$0 < \frac{\alpha}{a}\ \frac{1}{ak} < \frac{n}{2(ak)^2} - \frac{1}{2(n+1)} ;$$

otherwise it will have none.

It is clear that for sufficiently large n (fixed ak) there will always be such solutions. From the ordering theorem it follows that $v_n < v_{n+1}$ (for single solution cases). For the traveling wave guide application it is desirable to separate these solutions. As can be seen from Eq. (35), the functions $I_{\alpha,n}(v)$ separate for ak small relative to one. One obtains an approximate value for such solutions, valid for $\frac{\alpha}{a}\ \frac{1}{ak} < 1$, from the asymptotic expansion in Eq. (28). This gives

$$v_n \simeq \frac{a}{\alpha}\ ak\,(1 - \frac{n}{ak}) .$$

The corresponding wave numbers are

$$\beta_n \simeq k\left[1 + (1 - \frac{n}{ak})^2\,(\frac{a}{\alpha})^2\right]^{\frac{1}{2}} .$$

<u>5. The E-modes.</u> We have seen that a good way to isolate the zero-order I-mode in phase velocity from the other I-modes is to design the helical wave guide so that $\frac{\alpha}{a}\ \frac{1}{ak} < \frac{1}{2}$ and ak is small relative to one. We shall show in the present section that a wave guide built to such specifications can propagate no E-modes. We shall, in fact, find the set of all values of $\frac{\alpha}{a}\ \frac{1}{ak}$ and ak for which no E-modes can exist (see Fig. 2).

Aside from the zero-order mode, the function $z_{1,n}(u)$ has a pole at the origin and is a monotonic decreasing function (see Fig. 7). It approaches $-\infty$ from the left and $+\infty$ from the right at each of the zeros of $J_n(u)$. The only way to eliminate such a mode is to choose the parameters so that any intersection of the lines $y = \pm \frac{\alpha}{a} \frac{1}{ak}$ and the curve $y = z_{1,n}(u)$ lies to the right of the line $u = ak$. For the zero-order case a solution for $u > ak$ results in an attenuated mode, whereas for $n > 0$, $z_{1,n}(u)$ is complex valued for $u > ak$ so that there is no such solution. The situation for $z_{2,n}(u)$ is somewhat similar except that these functions are in general not as well behaved as the $z_{1,n}(u)$ functions. In this section the independent argument will always be u. We shall designate the k^{th} zero of $J_n(u)$ by x_k^n.

Lemma 5. z_n is a monotonically decreasing function of u.

As can be seen from the series expansion, Eq. (22a), $\frac{dz_n}{du}$ is negative for small values of u. Since the dominant term for large z_n on the right-hand side of Eq. (21a) is $-u\, z_n^2$, it follows that $\frac{dz_n}{du}$ is likewise negative in the neighborhood of all the poles of z_n. It remains to show that $\frac{dz_n}{du}$ stays negative between poles. If this were not so, $\frac{dz_n}{du}$ would have to vanish at a regular point. However, we find in the usual way that for such a point

$$\frac{d^2 z_n}{du^2} = -2z_n^2 - \frac{2n^2}{u^4} < 0.$$

This implies that $\frac{dz_n}{du}$ can be equal to zero only if it changes from positive to negative values, which is contrary to the fact that $\frac{dz_n}{du}$ starts out negative from all poles.

Lemma 6. $z_n < z_{n+1}$ for all u in the interval $(0, x_1^n)$.

Let $\Delta z = z_n - z_{n+1}$. It follows from the series expansion Eq. (22a), that for small u, Δz behaves like $-1/u^2$. By Eq. (21a),

$$\frac{d\Delta z}{du} = -u\,(z_n + z_{n+1})\,\Delta z - \frac{2}{u}\,\Delta z - \frac{2n+1}{u^3}.$$

Again, Δz starts out negative and cannot become zero in the interval of regularity $(0, x_1^n)$ since, if Δz should vanish in this interval, then at this point $\frac{d\Delta z}{du} = -\frac{2n+1}{u^3} < 0$, which implies that Δz changes from positive to negative values.

Theorem 5. $R_{1,n}$ is a monotonically decreasing function of u

$(0 < u < ak, \;\; n \geq 0)$.

By Eq. (19),

$$R_{1,n}(u) = \frac{n}{u^2}\left[1 - \left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} + z_n .$$

Hence $R_{1,n}(u)$ is the sum of two monotonically decreasing functions.

The series expansion for $R_{1,n}(u)$ is readily obtainable by use of Eq. (19) and (22a):

$$R_{1,n}(u) = \frac{2n}{u^2} - \left[\frac{1}{2(n+1)} + \frac{n}{2(ak)^2}\right] - - - .$$

We are again interested in solutions of the equation

$$\pm \frac{\alpha}{a}\,\frac{1}{ak} = R_{1,n}(u). \qquad (16c)$$

For $n = 0$, $R_{1,0}(u) = z_0$ starts with a value of $-1/2$ at $u = 0$, and decreases toward a pole at $x_1^0 = 2.405$. Hence if $ak < x_1^0$ one clearly avoids a solution of Eq. (16c) if and only if

$$\left.\begin{array}{l} \frac{\alpha}{a}\,\frac{1}{ak} < \frac{1}{2} \qquad (17a) \\ \quad \text{or} \;\; > -z_0(ak) \qquad (17b) \end{array}\right\} .$$

On the other hand, in the interval $(0, x_1^1)$ $R_{1,1}(u)$ starts at $+\infty$ and decreases monotonically to $-\infty$. To avoid a solution of (18c) in this interval ak must first of all lie in the interval.

Since the lowest point on $R_{1,1}(u)$ for $u \leq ak$ is precisely $R_{1,1}(ak) = z_1(ak)$, Eq. (18c) will have a solution for the plus sign unless

$$0 < \frac{\alpha}{a} \quad \frac{1}{ak} < z_1(ak), \tag{36}$$

where of course $0 < ak < x_1^1$. Eq. (36) further restricts ak since $z_1(ak)$ must be positive. The first zero of $z_1(ak)$ is at $x_z = 1.84$. Since $x_z < x_0^1$, it follows that solutions of Eq. (18c) for both n = 0 and n = 1 will be eliminated if conditions (17a,b) and (36) are both valid. We shall now show that these conditions are sufficient to eliminate any solution of Eq. (18c). Since $x_1^0 < x_1^1 < x_1^2 < - - -$, this is an immediate consequence of the following theorem.

<u>Theorem 4.</u> $R_{1,n}(u) < R_{1,n+1}(u)$ for all u in the interval $0, \min(ak_1 \; x_1^n)$.

Let $\Delta R_1 = R_{1,n} - R_{1,n+1}$. Then

$$\Delta R_1 = -\frac{1}{u^2}\left[1 - \left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} + \Delta z .$$

ΔR_1 is therefore the sum of two negative functions in that part of $(0, x_1^n)$ for which it is real valued.

In order to eliminate the modes associated with the $R_{2,n}(u)$ functions one must restrict the parameters $\frac{\alpha}{a}, \frac{1}{ak}$ and ak still further. Solutions of

$$\pm \frac{\alpha}{a} \quad \frac{1}{ak} = R_{2,n}(u) \tag{18d}$$

for n = 0 have already been considered since $R_{2,o}(u) = -R_{1,o}(u)$. The other functions $(n > 0)$ are not easily handled. As can be seen from the series expansion about the origin

$$R_{2,n}(u) = \left[\frac{1}{2(n+1)} - \frac{1}{2(ak)^2}\right] + \left[\frac{1}{8(n+1)(n+2)} - \frac{1}{8(ak)^4}\right] u^2 - - - ,$$

these functions depend rather strongly upon ak. We shall first consider the function $R_{2,1}(u)$ in detail. As we shall see, the modes associated with the $R_{2,n}(u)$ for $n > 1$ can be eliminated without a detailed analysis.

For convenience in notation and in order to show the k-dependence more clearly, let us designate $R_{2,1}(u)$ by $R(u,k)$. That is

$$R(u,k) = \frac{1}{u^2}\left[1 - \left(\frac{u}{ak}\right)^2\right]^{\frac{1}{2}} - z_1 .$$

We now prove two lemmas about the k-dependence.

<u>Lemma 7</u>. $R(u,k_1) < R(u,k_2)$ and $\frac{d}{du} R(u;k_1) < \frac{d}{du} R(u,k_2)$ for $k_1 < k_2$ and $u \leq ak_1$.

If we make the substitutions

$$r = \left[1 - \left(\frac{u}{ak_1}\right)^2\right]^{\frac{1}{2}}$$

and

$$s = \left[1 - \left(\frac{u}{ak_2}\right)^2\right]^{\frac{1}{2}} ,$$

then

$$R(u,k_1) - R(u,k_2) = \frac{1}{u^2}\left(r - s\right)$$

$$= \frac{k_1^2 - k_2^2}{(ak_1k_2)^2} \cdot \frac{1}{r + s} \qquad (37)$$

The first part of the lemma follows immediately from the fact that $k_1 < k_2$. Since r and s are both positive monotonic decreasing functions of u, $(r + s)^{-1}$ is monotonic increasing, and the right-hand side of Eq.(37) is therefore monotnoic decreasing for $k_1 < k_2$. Hence, under this condition, its derivative is negative, which proves the second part of the lemma.

Lemma 8. $R(u,k)$ is monotonic decreasing if $ak \leq \sqrt[4]{12}$ for all u in the interval $(0,ak)$.

It follows from the preceding lemma that we need only show that $R(u,k_0)$ is monotonic decreasing where $ak_0 = \sqrt[4]{12}$. The series expansion for this function is

$$R(u,k_0) = \left[\frac{1}{4} - \frac{1}{2(ak_0)^2}\right] + \left[\frac{1}{96} - \frac{1}{8(ak_0)^4}\right]u^2 + \left[\frac{1}{24 \times 64} - \frac{1}{16(ak_0)^6}\right]u^4 - -$$

$$= 0.106 - 0.0009u^4 - - -$$

It is clear that $R(u,k_0)$ is positive and decreasing for small values of u. Further, it is positive at the upper end of the interval since, by the definition of $R(u,k)$, $R(ak_0,k_0) = -z_1(ak_0) > 0$. (Note that $ak_0 = \sqrt[4]{12} = 1.861 > 1.84 = x_2$.)

The differential equation for $R(u,k_0)$ is

$$\frac{dR}{du} = uR^2 - \frac{2}{u}\left\{\left[1 - \left(\frac{u}{ak_0}\right)^2\right]^{\frac{1}{2}} + 1\right\}R + \frac{1}{u} \tag{38}$$

$$- \frac{1}{u}\,\frac{1}{(ak_0)^2}\;\frac{\left[1 - \left(\frac{u}{ak_0}\right)^2\right]^{\frac{1}{2}} + 1}{\left[1 - \left(\frac{u}{ak_0}\right)^2\right]^{\frac{1}{2}}}\;.$$

If $R(u,k_0)$ were ever negative its graph would have to cross the u-axis at least twice, the first time with a negative slope and the last time with a positive slope. However, for $R = 0$, we find that if b is approximately equal to 1.7, $\frac{dR}{du}$ is positive for $u < b$ and negative for $u > b$. Hence $R(u,k_0)$ could only cross below the u-axis for $u \geq b$ and it could not thereafter recross. Therefore $R(u,k_0) \geq 0$.

Suppose now that $\frac{dR}{du}$ vanishes for some u, say u_1 in the interval $(0,ak_0)$. Differentiating Eq. (38) and setting $\frac{dR}{du}$ equal to zero, we obtain

$$\left.\frac{d^2R}{du^2}\right]_{u_1} = R^2 + \frac{2}{(ak_0)^2}\;\frac{R}{\left[1 - \left(\frac{u_1}{ak_0}\right)^2\right]^{1/2}} + \frac{1}{(ak_0)^4}\;\frac{1}{\left[1 - \left(\frac{u_1}{ak_0}\right)^2\right]^{3/2}}$$

Since this expression is positive for any u_1 in the interval $(0, ak_0)$, it follows that F may have only local minima. If it had such a minimum it would have to approach the point $u = ak_0$ from below. This is impossible since it is clear from Eq. (38) that the slope of $R(u,k_0)$ approaches $-\infty$ as u approaches ak_0 from the left.

Theorem 7. In order that neither $R_{1,1}(u)$ nor $R_{2,1}(u)$ equal $\frac{\alpha}{a}\,\frac{1}{ak}$ for some u, it is necessary and sufficient that

$$0 < \frac{\alpha}{a}\,\frac{1}{ak} < \frac{1}{2(ak)^2} - \frac{1}{4} \,. \tag{39}$$

We have already shown that condition (36) must be satisfied in order to eliminate an $R_{1,1}$ solution. Here $0 < ak < x_s$. If $\sqrt{2} < ak < x_s$, then $R_{2,1}(u)$ starts positive and decreases monotonically to the value $-z_1(ak) < 0$. Hence there will be an $R_{2,1}$ solution in this case if $\frac{\alpha}{a}\,\frac{1}{ak} < z_1(ak)$. Thus for ak in this range there will always be either an $R_{1,1}$ solution or an $R_{2,1}$ solution. Finally if $ak < \sqrt{2}$, $R_{2,1}(u)$ starts at the value $\frac{1}{4} - \frac{1}{2(ak)^2} < 0$ and decreases monotonically to the value $-z_1(ak) < 0$. It follows that the only way to avoid both an $R_{1,1}$ and an $R_{2,1}$ solution is to choose $\frac{\alpha}{a}\,\frac{1}{ak}$ in the range given by (39).

In order, therefore, to eliminate both the zero-order R-modes and the first-order R-modes it is necessary and sufficient that conditions (17a, b) and (39) be satisfied. The region of values of the parameters $\frac{\alpha}{a}\,\frac{1}{ak}$ and ak which satisfy both conditions (17a,b) and (39) is shown in Fig. 2. Because of the following theorem the other R-modes are also eliminated if these conditions are satisfied. For, just as the $R_{1,n}$ functions $(n > 1)$

were above $R_{1,1}$ in $(0,x_1^1)$, so the $R_{2,n}$ functions $(n > 1)$ are below $R_{2,1}$ in $(0,x_1^1)$ and can furnish no solutions of (18d) if condition (39) is satisfied.

Theorem 8. $R_{2,n}(u) > R_{2,n+1}(u)$ for all u in the interval $\left[0, \min(ak, x_1^n)\right]$.

We prove this theorem by means of the usual lemmas. We first introduce the function

$$w_n = \frac{n}{u^2} - z_n \ .$$

The expansion for w_n about the origin is

$$w_n = \frac{1}{2(n+1)} + \frac{1}{8(n+1)^2\,(n+2)} \quad u^2 \quad - -$$

Lemma 9. w_n is a monotonic increasing function of u, positive in the interval $(0,x_1^n)$.

From the series expansion for w_n, it is clear that it is positive for small values of u. The differential equation for w_n is

$$\frac{dw_n}{du} = u\,w_n^2 - \frac{2(n+1)}{u}\,w_n + \frac{1}{u} \tag{40}$$

It follows from the usual argument that in the first inerval in which it is regular, $(0, x_1^n)$, w_n must remain positive. It is likewise clear from Eq. (40) that $\frac{dw_n}{du}$ is positive in the neighborhood of the poles, and from the expansion that it is positive for small positive u. By studying the second derivative one shows in the usual way that $\frac{dw_n}{du}$ must remain positive between these singularities.

<u>Lemma 10.</u> $w_n > w_{n+1}$ for all u in the interval $(0, x_1^n)$.

Again define $\Delta w = w_n - w_{n+1}$. For $u = 0$, $\Delta w = \dfrac{1}{2(n+1)(n+2)}$.

Furthermore

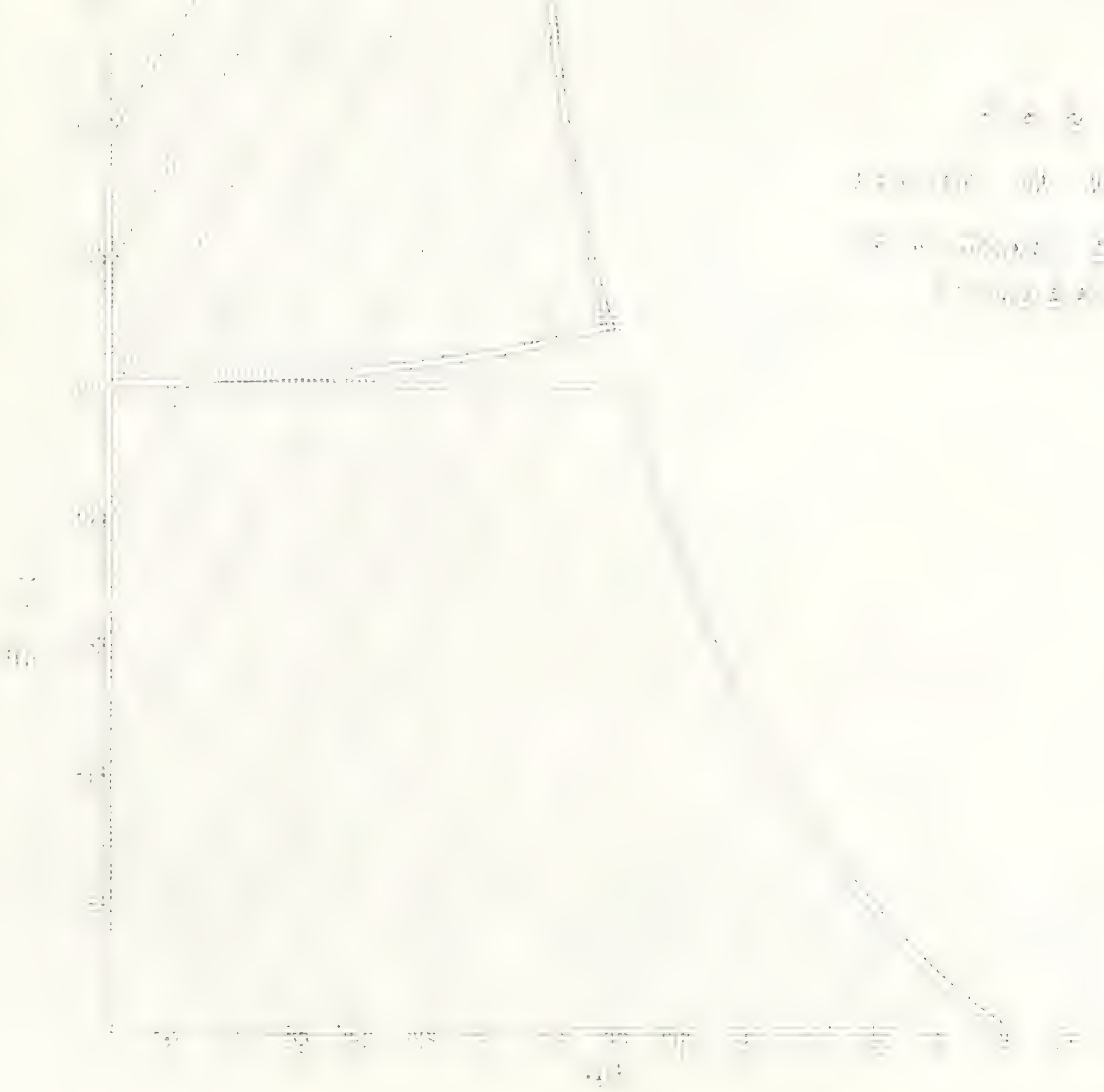
$$\frac{d\,\Delta w}{du} = u(w_n + w_{n+1})\Delta w - \frac{2}{u}(n+2)\Delta w + 2w_n.$$

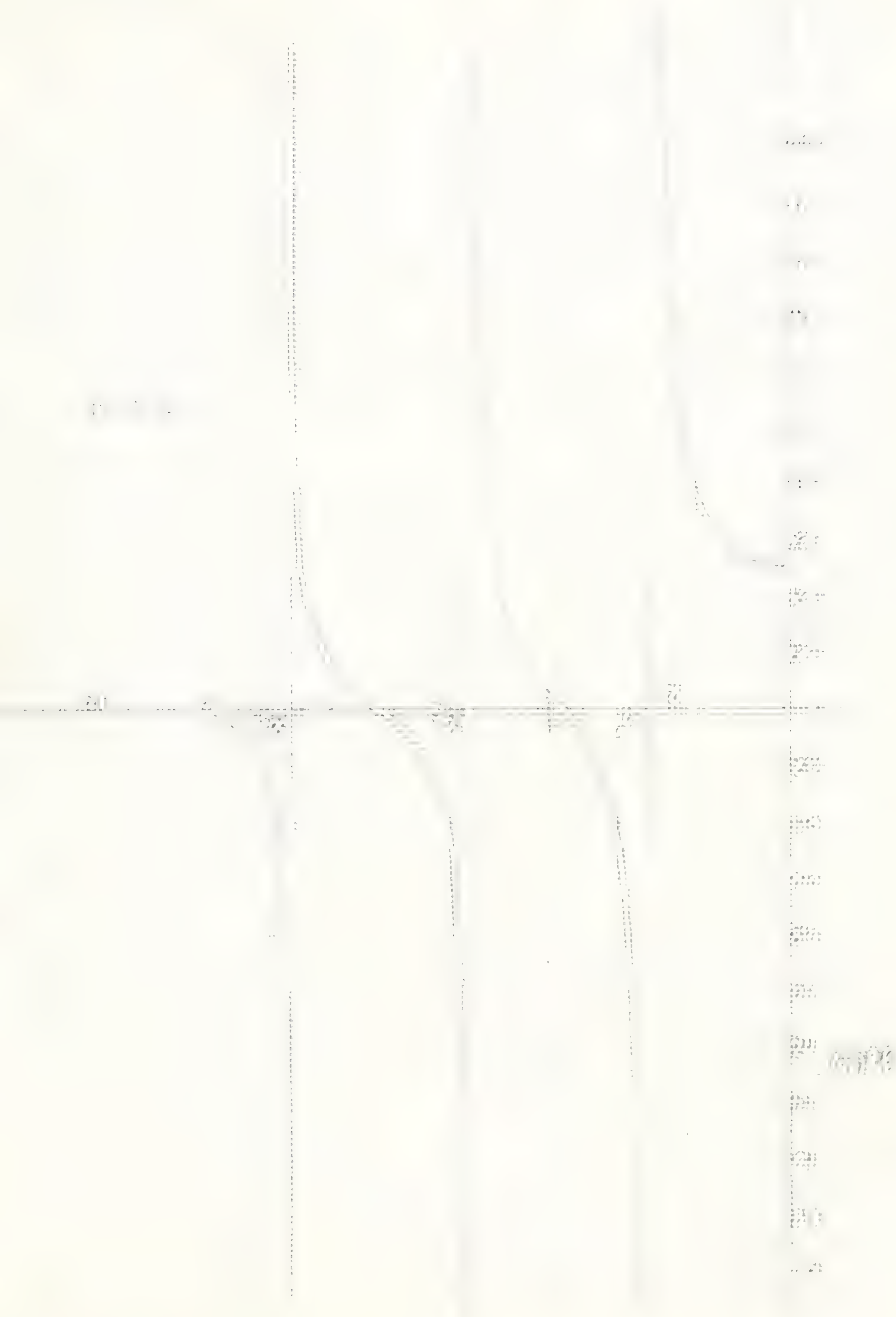
Again it follows that Δw must remain positive in the interval in which it is regular.

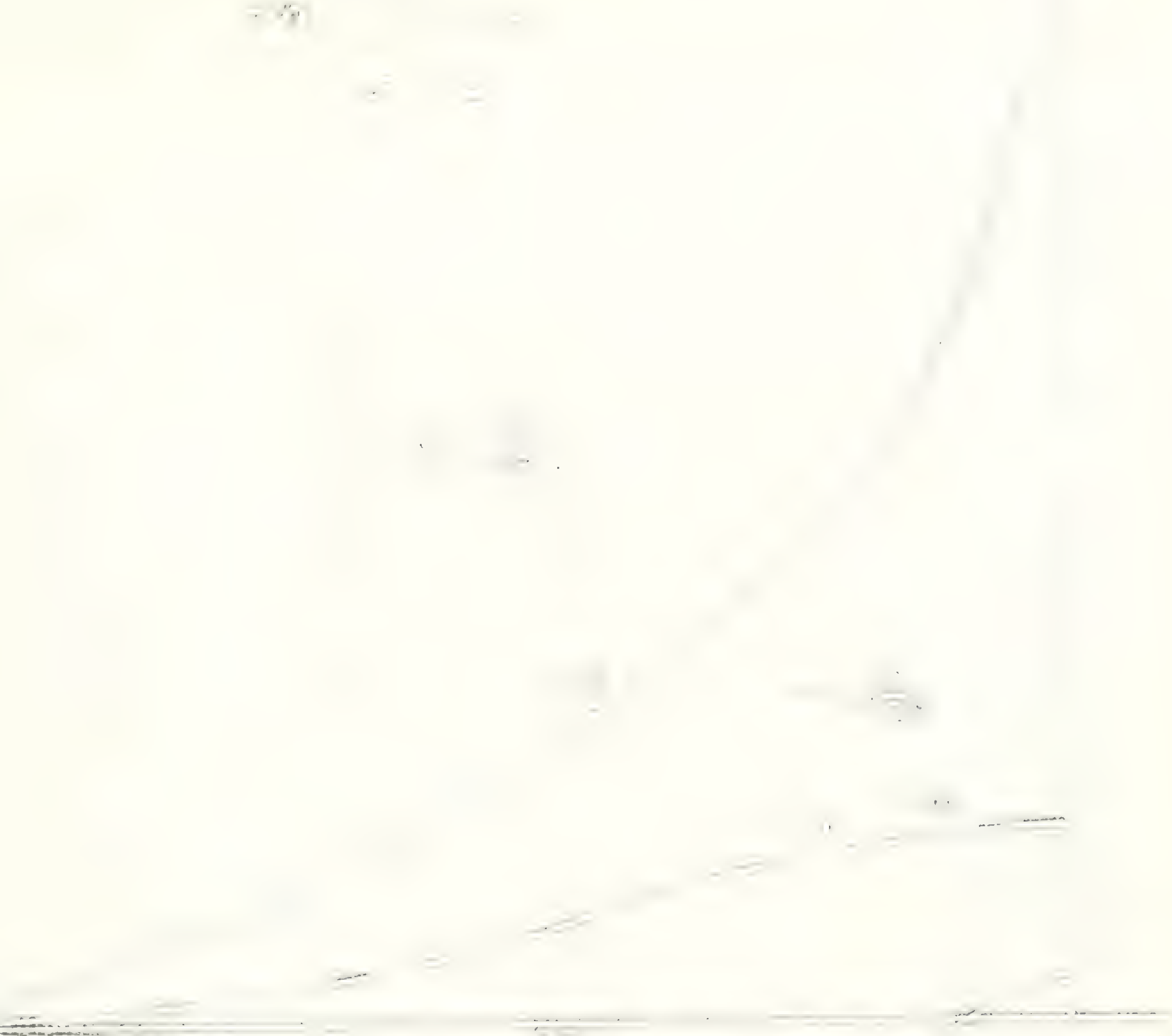
Finally

$$R_{2,n}(u) - R_{2,\,n+1}(u) = \frac{1}{u^2}\left\{1 - \left[1 - \left(\frac{u}{nk}\right)^2\right]^{\frac{1}{2}}\right\} + \Delta w$$

The first term is positive for all u in the interval (0, nk) and, by lemma 9, the second is positive in the interval $(0, x_1^n)$, which proves theorem 5.







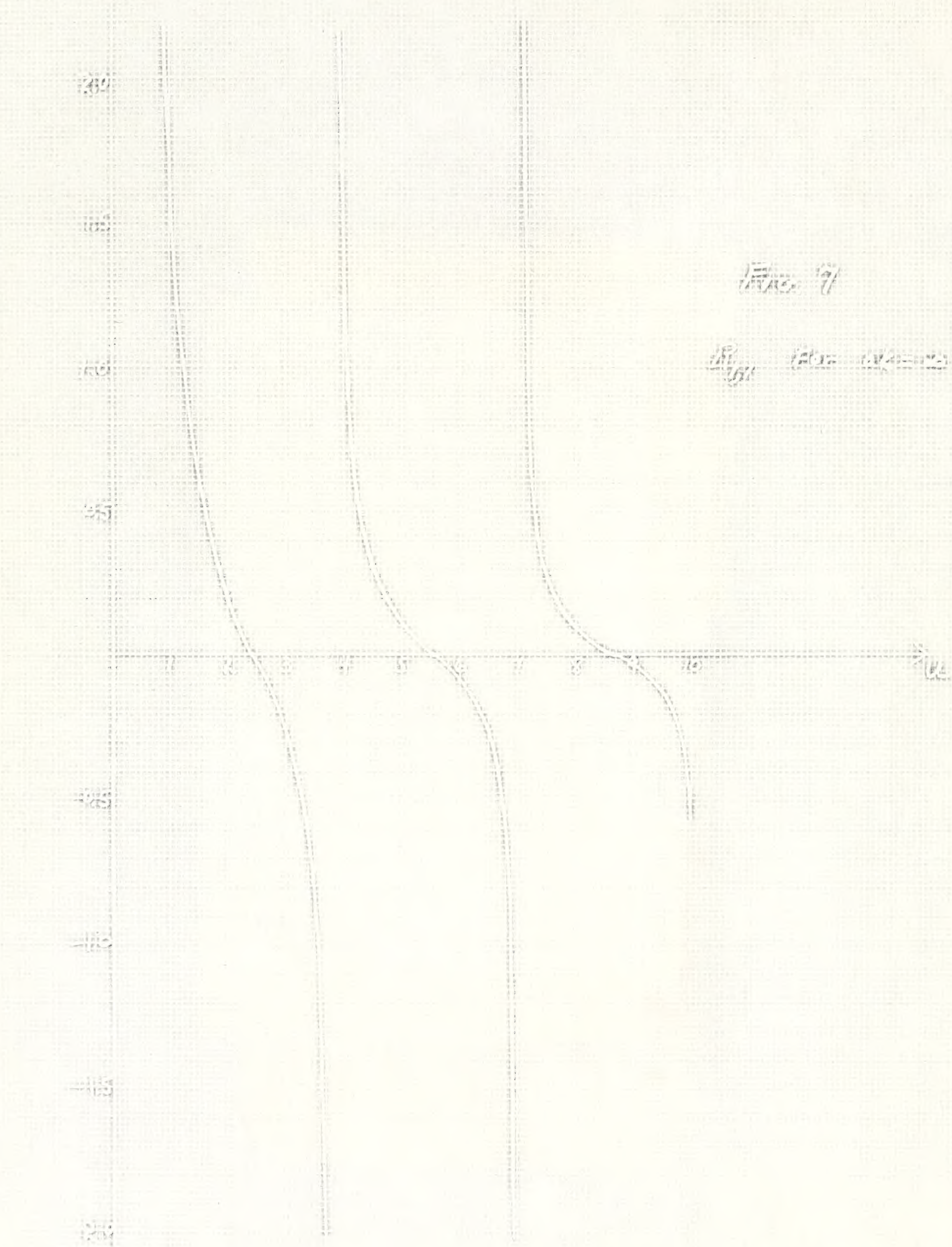


Fig. 7